DYNAMIC THREE-DIMENSIONAL EQUATIONS OF THE RAKHMATULIN ELASTIC – PLASTIC MODEL

E. I. Romenskii

UDC 539.374

This paper is devoted to an extension of the Rakhmatulin equations for elastic–plastic wave propagation [1, 2] to the three-dimensional case. Rules are formulated for the construction of the dynamics equations of a continuous medium which can be used for some models of inelastic media. These rules are obtained by formalization of the ideas contained in [3], which were relied upon for writing the equations of nonlinear viscoelasticity on the basis of the Maxwell relaxation model. The basic idea is to use the concept of the effective elastic strain (strain of elastic unloading of an element of the medium) to describe the stress state. The tensor of the actual distortion is thereby represented in the form of the product of the effective elastic and plastic distortion tensors. The description of inelastic strains in the models under consideration consists of giving the law for the rate of change in the plastic strain as a function of the stress state of the medium and of its rate of change.

The system of dynamics equations consists of the momentum and energy conservation laws as well as of the equations for the evolution of the effective elastic distortion tensor or of some effective elastic strain tensor. The Hencky logarithmic strain tensor is selected here. The law governing the behavior of the medium under plastic strains is expressed as follows: The rate of change of the plastic strains in each direction is represented in the form of a linear combination of the rates of change of the stresses with coefficients dependent on the state of the medium. In substance, this extends the method of determining the plastic strains in the one-dimensional Rakhmatulin model. The main requirement on the selection of the mentioned combination of the rates of change of the stresses is the correctness of the system of dynamical equations obtained. The correctness requirement permits the unique selection of the coefficients needed in this linear combination. The investigation is performed for the equation of state (internal energy density) of the form $E = E^0(\rho, S) + E^1(h_1, h_2, h_3)$, where ρ is the density, h_i are the principal values of the effective elastic Hencky strain tensor, and S is the entropy.

The possibility of applying the dynamical equations formulated is shown in a simple example of homogeneous strain of a flat layer.

1. EQUATIONS OF MOTION OF A CONTINUOUS MEDIUM

Let us write the equations of motion of a continuous medium in Euler coordinates by using the concept of an effective elastic strain, the strain of elastic unloading of an element of the medium. The distortion tensor is used as a measure of the strain. A standard stress-free state, determined by using the effective elastic distortion in a known stress state, is introduced for each state of the element of the medium. On the basis of this matrix, the total distortion is separated into the product of the elastic and plastic distortion matrices. The selection of formulas to calculate the plastic distortion is determined as a function of the model of the inelastic medium for which the dynamical equations must be written out.

Let us formulate the equations of motion of a deformable medium without taking account of the internal processes occurring therein (we call internal friction, heat transfer, chemical reactions between medium constituents, etc. such processes). We shall characterize the state of the medium at each point of space provided with a Cartesian coordinate system x_i by the following "observable" quantities at the time t: the velocity field $u_i(t, x_1, x_2, x_3)$, the stress field $\sigma_{ij}(t, x_1, x_2, x_3)$, and the temperature $T(t, x_1, x_2, x_3)$. We shall consider the momentum and energy conservation laws, which are described well by the known equations

$$\rho du_i/dt - \partial\sigma_{ij}/\partial x_j = 0; \quad (1.1)$$

$$\rho dE/dt - \sigma_{ij}\partial u_i/\partial x_j = 0, \quad (1.2)$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 138-158, March-April, 1979. Original article submitted May 19, 1978.

to be satisfied during motion of the medium, where $d/dt = \partial/\partial t + u_\alpha \partial/\partial x_\alpha$ is the derivative along the motion trajectory $\rho(t, x_1, x_2, x_3)$ is the density of the medium, and $E(t, x_1, x_2, x_3)$ is the internal energy density (we indicate its relationship to the stress state of the medium below).

It is known that the properties of a medium often depend on what strains occurred prior to the running time t. The "distortion" of its behavior must thereby be known for the correct description of the medium. Let observation of the motion of the medium start at the time $t = 0$; i.e., the coordinates are known for the particle x_i^0, which has the coordinate x_i at a given time t. The motion of this particle is given by the functions $x_i = x_i(t, x_1^0, x_2^0, x_3^0)$, which can be obtained by solving the equations

$$dx_i(t)/dt = u_i(t, x_1, x_2, x_3), \quad x_i(0) = x_i^0. \tag{1.3}$$

Let us consider the functions $x_i(t, x_1^0, x_2^0, x_3^0)$ continuous and mutually one-to-one; the dependence $x_i^0(t, x_1, x_2, x_3)$ can thereby be determined. It turns out to be convenient to describe the strain on the medium by the matrix of the Jacobian $\partial x_i^0/\partial x_j$. The conjugate matrix to this Jacobian: $a_{ij}^0 = \partial x_j^0/\partial x_i$, is called the real (total) distortion matrix. By using formal differentiation, equations for the real distortion are obtained from (1.3):

$$\frac{da_{ij}^0}{dt} + \frac{\partial u_\alpha}{\partial x_i} a_{\alpha j}^0 = 0$$

or in matrix form for $A^0 = \|a_{ij}^0\|$ and $W = \left\| \frac{\partial u_i}{\partial x_j} \right\|$,

$$dA^0/dt + W^* A^0 = 0. \tag{1.4}$$

However, the real distortions are not associated with the stress state of the medium. Indeed, although real strains can also not occur in a medium, the stress field therein can change because of relaxation or other inner processes. The so-called effective elastic strain [3] [this is the strain which must be executed elastically (by adiabatic means) on the element of the medium in order to transfer it from the stress state to the state with a given stress field] is related to the stress state.

Let a particle of the medium be at a point x_i at the time t and, after unloading, the element of the medium containing this particle is at the point $\bar{x}_i$. The matrix $a_{ij}^e = \partial \bar{x}_j/\partial x_i$ is also called the effective elastic distortion matrix. Let us note that, as a rule, it is impossible to introduce the coordinates $\bar{x}_i$ for the medium as a whole; i.e., it is impossible to liberate a stressed body elastically from the stresses so that it would be entirely within the Euclidean space. However, such a procedure can be executed in the small neighborhood of a single particle, where the coordinates are defined just to the accuracy of a rotation [3].

Using the effective elastic distortion concept introduced, we obtain

$$a_{ij}^0 = \frac{\partial x_j^0}{\partial x_i} = \frac{\partial x_j^0}{\partial \bar{x}_\alpha} \frac{\partial \bar{x}_\alpha}{\partial x_i} = a_{\alpha j}^p a_{i\alpha}^e = a_{i\alpha}^e a_{\alpha j}^p \tag{1.5}$$

or in matrix form

$$A^0 = A^e A^p. \tag{1.6}$$

A new object has been introduced, the matrix $A^p = \|a_{ij}^p\|$, which we designate the plastic (residual) distortion matrix. It characterizes the irreversible strain, i.e., the strains which remain in the medium after unloading. From (1.4) and (1.6) we obtain

$$\frac{dA^e}{dt} + W^* A^e = -A^e \frac{dA^p}{dt} (A^p)^{-1}.$$

Let $\dot{\Phi} = \|\dot{\varphi}_{ij}\| = -(dA^p/dt)(A^p)^{-1}$ (the dot does not denote differentiation here; this is simply a symbolic notation). Furthermore, the superscript e will be omitted exactly as the word "effective" in the notation for the effective elastic distortion. Therefore, the following equations for the evolution of the elastic distortion are obtained:

$$dA/dt + W^* A = A\dot{\Phi} \tag{1.7}$$

and in the component-by-component form

$$\frac{da_{ij}}{dt} + \frac{\partial u_\alpha}{\partial x_i} a_{\alpha j} = a_{i\alpha} \dot{\varphi}_{\alpha j}. \tag{1.8}$$

The matrix A admits the representation

$$A = UK^{-1}V, \tag{1.9}$$

where U, V are orthogonal matrices, the matrix K^{-1} is a diagonal matrix

$$K^{-1} = \begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & k_3^{-1} \end{pmatrix},$$

and the k_i have the meaning of tension (compression) factors along the principal elastic strain axes.

The stress tensor has the form

$$\|\sigma_{ij}\| = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} U^*,$$

where $\sigma_i = \rho k_i E_{k_i}$ are the principal stresses, $\rho = \rho_0/k_1k_2k_3$ is the density, $E(k_1, k_2, k_3, S)$ is the internal energy density, and S is the entropy. The stress is calculated directly in terms of the distortion tensor by means of the formulas

$$\|\sigma_{ij}\| = \|-\rho a_{i\alpha}E_{a_{j\alpha}}\| = -\rho A\partial E/\partial A^*.$$

Let us note that the temperature of the medium is calculated by the formula $T = \partial E/\partial S(k_1, k_2, k_3, S)$.

Thus, the system of equations

$$\rho\, du_i/dt - \partial\sigma_{ij}/\partial x_j = 0, \quad \rho dE/dt - \sigma_{ij}\partial u_i/\partial x_j = 0,$$
$$da_{ij}/dt + a_{\alpha j}\partial u_\alpha/\partial x_i = a_{i\alpha}\dot{\varphi}_{\alpha j}, \tag{1.10}$$
$$\rho = \rho_0 \det\|a_{ij}\|, \quad \sigma_{ij} = -\rho a_{i\alpha}E_{a_{j\alpha}}, \quad E = E(k_1, k_2, k_3, S)$$

is obtained to describe the motion of a continuous medium.

To close this system of equations, it is necessary to give the terms $\dot{\varphi}_{ij}$ describing the inelastic strains, by some method. It turns out to be convenient to relate $\dot{\varphi}_{ij}$ to the existing stress state of the medium and to its temperature; thereby $\dot{\varphi}_{ij}$ is determined in terms of the elastic strains and the entropy. This relation is established in [3] by using interpolation by the Maxwell relaxation terms which postulate stress relaxation to a global tensor. A method will be given in this paper for giving $\dot{\varphi}_{ij}$ by using an elastic–plastic scheme.

Now, let us indicate certain corollaries of the system (1.10). Let us recall that a_{ij} is the effective elastic distortion. An equation for the density $\rho = \rho_0 \det\|a_{ij}\|$ results from the equations for a_{ij}. To do this the equation for a_{ij} must be multiplied correspondingly by $\rho_{a_{ij}} = \rho c_{ji}$, where c_{ji} are elements of the inverse matrix to $A = \|a_{ij}\|$, $C = A^{-1}$, and all the equations are added, after which we obtain

$$\rho_{a_{ij}}\left(\frac{da_{ij}}{dt} + a_{\alpha j}\frac{\partial u_\alpha}{\partial x_i}\right) = \rho_{a_{ij}}a_{i\alpha}\dot{\varphi}_{\alpha j}, \quad \frac{d\rho}{dt} + \rho c_{ji}\, a_{\alpha j}\frac{\partial u_\alpha}{\partial x_i} = \rho c_{ji}a_{i\alpha}\dot{\varphi}_{\alpha j},$$

and by using the identities $c_{ji}a_{\alpha j} = \delta_{i\alpha}$, $c_{ji}a_{i\alpha} = \delta_{j\alpha}$,

$$d\rho/dt + \rho\partial u_i/\partial x_i = \rho\dot{\varphi}_{jj}.$$

An equation has therefore been obtained for the density with the right side $\rho(\dot{\varphi}_{11} + \dot{\varphi}_{22} + \dot{\varphi}_{33})$. Therefore, for compliance with the mass conservation law it is necessary to require that

$$\dot{\varphi}_{11} + \dot{\varphi}_{22} + \dot{\varphi}_{33} = 0. \tag{1.11}$$

This equality denotes compliance with the incompressibility condition for the plastic strains (the models under consideration do not take account of the volume compressibility of the medium).

A corollary to the system (1.10) is the equation for the entropy. In fact, by using the dependence $E = E(a_{11}, a_{12}, \ldots, a_{33}, S)$ we have

$$\frac{dE}{dt} = E_{a_{ij}}\frac{da_{ij}}{dt} + E_S\frac{dS}{dt} = \frac{1}{\rho}\sigma_{ij}\frac{\partial u_i}{\partial x_j},$$

from which

$$\frac{dS}{dt} = -\frac{1}{E_S} E_{a_{ij}} \frac{da_{ij}}{dt} + \frac{1}{\rho E_S} \sigma_{ij} \frac{\partial u_i}{\partial x_j} = \frac{1}{E_S} E_{a_{ij}} a_{\alpha j} \frac{\partial u_\alpha}{\partial x_i} - \frac{1}{E_S} E_{a_{ij}} a_{i\alpha} \dot{\varphi}_{\alpha j} + \frac{1}{\rho E_S} \sigma_{ij} \frac{\partial u_i}{\partial x_j}.$$

Now using the expression for $\sigma_{ij} = -\rho a_{i\alpha} E_{a_{j\alpha}}$, we see that the terms with derivatives of the velocities cancel and we obtain

$$\frac{dS}{dt} = -\frac{1}{E_S} E_{a_{ij}} a_{i\alpha} \dot{\varphi}_{\alpha j}.$$

Let us note that $E_{a_{ij}} a_{i\alpha} \dot{\varphi}_{\alpha j} = tr\left(\frac{\partial E}{\partial A} \dot{\Phi}^* A^*\right) = tr\left(A\dot{\Phi} CA \frac{\partial E}{\partial A^*}\right)$, where $C = A^{-1}$, and we obtain by using $\|\sigma_{ij}\| = -\rho A \cdot \partial E / \partial A^*$,

$$\frac{dS}{dt} = \frac{1}{\rho E_S} tr(A\dot{\Phi} C \|\sigma_{ij}\|) = \frac{1}{\rho E_S} a_{i\alpha} \dot{\varphi}_{\alpha\beta} c_{\beta\gamma} \sigma_{\gamma i}, \tag{1.12}$$

$$C = \|c_{ij}\| = A^{-1}.$$

As is known, the second law of thermodynamics dictates no decrease in the entropy, which implies the need to comply with the inequality

$$a_{i\alpha} \dot{\varphi}_{\alpha\beta}\, c_{\beta\gamma} \sigma_{\gamma i} \geqslant 0. \tag{1.13}$$

Therefore, constraints (1.11) and (1.13) have been obtained, which should be imposed on the selection of the terms taking account of the inelastic strains $\dot{\varphi}_{ij}$:

$$\dot{\varphi}_{11} + \dot{\varphi}_{22} + \dot{\varphi}_{33} = 0, \quad a_{i\alpha} \dot{\varphi}_{\alpha\beta} c_{\beta\gamma} \sigma_{\gamma i} \geqslant 0.$$

2. EQUATIONS FOR A LOGARITHMIC STRAIN TENSOR

We shall use the Hencky logarithmic strain tensor to formulate the dynamics equations of inelastic media. We shall write the equations in terms of the effective elastic strains which were introduced in Sec. 1.

The elastic distortion matrix A can be represented in the form of (1.9). It is convenient to use the metric strain tensor $G = AA^*$ to describe the strain, thereby getting rid of the matrix V: $G = AA^* = UK^{-2}U^*$. An equation for G can be obtained from (1.7) for A:

$$dG/dt + GW + W^*G = A(\dot{\Phi} + \dot{\Phi}^*)A^*. \tag{2.1}$$

Let

$$D_G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} = K^{-2},\ D_A = K^{-1}\ (G = UD_G U^*,\ A = UD_A V).$$

It is convenient to use the parameter $h_i = -(1/2)\ln g_i = \ln k_i$ to describe the strain in the principal axes. Indeed, we assume that at the time $t = 0$ all the strains have occurred only in the form of tension (compression) along the coordinate axes under the effect of the stress σ_i acting along these axes. In this case the elastic and plastic distortion matrices as well as the velocity gradient matrix will be diagonals. Equations (1.8) for the evolution of distortion have the form

$$\frac{dk_i^{-1}}{dt} + \frac{\partial u_i}{\partial x_i} k_i^{-1} = -k_i^{-1} \frac{d\,(k_i^p)^{-1}}{dt} k_i^p,$$

from which

$$\frac{dk_i}{dt} - k_i \frac{\partial u_i}{\partial x_i} = -k_i \frac{dk_i^p}{dt} (k_i^p)^{-1}.$$

Taking into account that $(k_i^p)^{-1} dk_i^p = d \ln k_i^p = dh_i^p$, we obtain for $h_i = \ln k_i$

$$\frac{dh_i}{dt} - \frac{\partial u_i}{\partial x_i} = -\frac{dh_i^p}{dt} = -\dot{\varphi}_i. \tag{2.2}$$

It is seen that $h_i + h_i^p$ have the meaning of real strains along the principal axes of the strain tensor.

Now, let us define the logarithmic strain tensor for arbitrary strains by means of the formula

$$H = UD_H U^*,$$

where $D_H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$; $h_i = -\frac{1}{2}\ln g_i = \ln k_i$; and U is the same matrix as in (1.9). Let us write the equation for the time variation of H. Using (2.1) we obtain

$$U\frac{dD_G}{dt}U^* + \frac{dU}{dt}D_G U^* - UD_G U^*\frac{dU}{dt}U^* + UD_G U^* W + W^* UD_G U^* = UD_A V(\dot{\Phi} + \dot{\Phi}^*)V^* D_A U^*.$$

Multiplying both sides of this equation on the left by U^*, on the right by U, and using the notation $\Omega = \|\omega_{ij}\| = U^* W U$, we obtain

$$\frac{dD_G}{dt} + U^*\frac{dU}{dt}D_G - D_G U^*\frac{dU}{dt} + D_G\Omega + \Omega^* D_G = 2D_A \Psi D_A, \tag{2.3}$$

where $\Psi = (1/2)V(\dot{\Phi} + \dot{\Phi}^*)V^* = \|\psi_{ij}\|$. It can be seen that zeros are on the diagonals for the matrix $D_G - D_G U^*\frac{dU}{dt}$ and the matrix dD_G/dt is a diagonal matrix. Writing the matrix equation (2.3) in component terms, we obtain an equation for the principal values of the tensor g_i:

$$dg_i/dt + 2g_i\omega_{ii} = 2g_i\psi_{ii},$$

and an equation for the nondiagonal elements of the matrix $U^* dU/dt$,

$$\left[U^*\frac{dU}{dt}\right]_{ij} - \frac{1}{g_i - g_j}(g_i\omega_{ij} + g_j\omega_{ji}) = -\frac{2\sqrt{g_i g_j}}{g_i - g_j}\psi_{ij}, \quad i \neq j. \tag{2.4}$$

Equations for $h_i = -(1/2)\ln g_i$ follow from the equations for g_i;

$$dh_i/dt - \omega_{ii} = -\psi_{ii}. \tag{2.5}$$

From the equation

$$U^*\frac{dH}{dt}U = \frac{dD_H}{dt} + U^*\frac{dU}{dt}D_H - D_H U^*\frac{dU}{dt}$$

and (2.4) and (2.5) there follows

$$\frac{dH}{dt} = U\begin{pmatrix} \omega_{11} & -\frac{h_1 - h_2}{g_1 - g_2}(g_1\omega_{12} + g_2\omega_{21}) & -\frac{h_1 - h_3}{g_1 - g_3}(g_1\omega_{13} + g_3\omega_{31}) \\ -\frac{h_2 - h_1}{g_2 - g_1}(g_2\omega_{21} + g_1\omega_{12}) & \omega_{22} & -\frac{h_2 - h_3}{g_2 - g_3}(g_2\omega_{23} + g_3\omega_{32}) \\ -\frac{h_3 - h_1}{g_3 - g_1}(g_3\omega_{31} + g_1\omega_{13}) & -\frac{h_3 - h_2}{g_3 - g_2}(g_3\omega_{32} + g_2\omega_{23}) & \omega_{33} \end{pmatrix}U^* -$$

$$- U\begin{pmatrix} \psi_{11} & -\frac{h_1 - h_2}{g_1 - g_2}2\sqrt{g_1 g_2}\psi_{12} & -\frac{h_1 - h_3}{g_1 - g_3}2\sqrt{g_1 g_3}\psi_{13} \\ -\frac{h_2 - h_1}{g_2 - g_1}2\sqrt{g_2 g_1}\psi_{21} & \psi_{22} & -\frac{h_2 - h_3}{g_2 - g_3}2\sqrt{g_2 g_3}\psi_{23} \\ -\frac{h_3 - h_1}{g_3 - g_1}2\sqrt{g_3 g_1}\psi_{31} & -\frac{h_3 - h_2}{g_3 - g_2}2\sqrt{g_3 g_2}\psi_{32} & \psi_{33} \end{pmatrix}U^*. \tag{2.6}$$

Here $g_i = e^{-2h_i}$; $\Omega = \|\omega_{ij}\| = U^*\|\partial u_n/\partial x_m\|U$.

Thus, equations have been obtained for the time variations in the logarithmic strain tensor H. These equations are not closed in the component h_{ij} since the right side of the equation is written in terms of the principal values h_i and the elements of the orthogonal matrix U; nevertheless the use of these equations is not a difficulty since h_i and U are determined from the equation

$$U\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}U^* = H.$$

The equations obtained for the tensor H are convenient for the construction of the dynamical equations of inelastic continuous media. Let us note that the formulas to calculate the stresses in terms of h_{ij} become simpler; it can be shown that

$$\sigma_{ij} = \rho \frac{\partial E}{\partial h_{ij}}, \text{ where } E = E(h_1, h_2, h_3, S); \quad \rho = \rho_0 e^{-(h_1+h_2+h_3)}, \tag{2.7}$$

where h_i are the principal values of the tensor h_{ij}.

The evolution equations for the tensor H in combination with (1.1) and (1.2) for the velocity and internal energy form a closed system if the method of calculating ψ_{ij}, the terms in the equations for h_{ij} which describe the plastic strains, is known.

Let us write the equation of the Maxwell relaxation model by using (2.6) for the logarithmic tensor H. We shall consider rotation of an element of the medium as a whole not to be accompanied by plastic strains; this means that

$$\psi_{ij} \equiv 0 \quad (i \neq j).$$

We take the following formula for ψ_{ii}:

$$\psi_{ii} = \frac{1}{\tau}\left(h_i - \frac{h_1 + h_2 + h_3}{3}\right),$$

where $\tau = \tau(h_1, h_2, h_3, S)$ is the characteristic relaxation time for the tangential stresses dependent on the state of the medium.

The selection of such formulas for the inelastic strains governs the relaxation of the effective elastic strain tensor to a global tensor, meaning (if a one-to-one connection between the stress and the effective elastic strain is assumed) the relaxation of the stress tensor to a global tensor.

3. DYNAMICAL EQUATIONS OF ELASTIC – PLASTIC STRAINS

Let us describe still another method of closing the system of equations of medium motion by using the selection of interpolation formulas for ψ_{ij}, the terms in the equations for the logarithmic tensor h_{ij} of the effective elastic strains which describe the rate of change on the inelastic strains.

Let us consider the case of strains occurring only along the axes x_i. The equations for the tensor h_{ij}, which will be diagonal, will hence have the form (2.2).

Let us select the invariant

$$h = (1/\sqrt{2})[(h_{11} - h_{22})^2 + (h_{22} - h_{33})^2 + (h_{33} - h_{11})^2 + 6(h_{12}h_{21} + h_{23}h_{32} + h_{31}h_{13})]^{1/2}, \tag{3.1}$$

which we call the tangential strain intensity, as the measure of the tangential strain. Let us note that by virtue of the invariance

$$h = (1/\sqrt{2})[(h_1 - h_2)^2 + (h_2 - h_3)^2 + (h_3 - h_1)^2]^{1/2}, \tag{3.2}$$

where h_i are the principal values of the tensor h_{ij}; i.e., h is independent of rotations of elements of the medium characterized by the matrix U.

Let us assume that the medium has the yield stress characterized by the magnitude of the tangential strains $h = h_*$. Let us consider that

$$\dot{\varphi}_i = \frac{dh_i^p}{dt} = 0 \text{ for } h < h_*, \quad \dot{\varphi}_i = \frac{dh_i^p}{dt} = \gamma_{ij}\frac{dh_j}{dt} \text{ for } h > h_*. \tag{3.3}$$

The coefficients γ_{ij} govern the change in the effective elastic strains under plastic deformation. Let us note that $\gamma_{ij} = 0$ can even be for $h > h_*$. This should hold for unloading.

Thus, in the case of triaxial strain without rotations, equations for h_i are postulated by using (3.3):

$$\frac{dh_i}{dt} - \frac{\partial u_i}{\partial x_i} = -\gamma_{ij}\frac{dh_j}{dt}. \tag{3.4}$$

Let us solve (3.4) for dh_i/dt. Using the notation $\|\beta_{ij}\| = \|\delta_{ij} + \gamma_{ij}\|^{-1}$, we obtain

$$\frac{dh_i}{dt} - \sum_{j=1}^{3} \beta_{ij} \frac{\partial u_j}{\partial x_j} = 0. \tag{3.5}$$

Let us make the selection of the matrix β_{ij} specific. First, we obtain three algebraic relationships for β_{ij} from the mass conservation law.

The mass conservation law is determined by the equation

$$\frac{d(h_1 + h_2 + h_3)}{dt} - \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) = 0. \tag{3.6}$$

From (3.5) we have

$$\frac{d(h_1 + h_2 + h_3)}{dt} - \sum_{j=1}^{3} (\beta_{1j} + \beta_{2j} + \beta_{3j}) \frac{\partial u_j}{\partial x_j} = 0,$$

from which it follows that by virtue of (3.6)

$$\beta_{1j} + \beta_{2j} + \beta_{3j} = 1.$$

Moreover, we assume isotropy of the plastic medium of the following nature: $\partial u_2/\partial x_2$ exerts the same influence on the change in h_1 as does $\partial u_3/\partial x_3$ (the same assumptions for h_2 and h_3). These requirements mean that

$$\beta_{12} = \beta_{13},\ \beta_{21} = \beta_{23},\ \beta_{31} = \beta_{32}.$$

Selecting three independent parameters,

$$l_1 = \beta_{12} = \beta_{13},\ l_2 = \beta_{21} = \beta_{31},\ l_3 = \beta_{31} = \beta_{32}, \tag{3.7}$$

we obtain

$$\beta_{11} = 1 - l_2 - l_3,\ \beta_{22} = 1 - l_3 - l_1,\ \beta_{33} = 1 - l_1 - l_2. \tag{3.8}$$

The equations for h_i take the following form:

$$\begin{aligned} \frac{dh_1}{dt} &= (1 - l_2 - l_3)\frac{\partial u_1}{\partial x_1} + l_1\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right), \\ \frac{dh_2}{dt} &= (1 - l_3 - l_1)\frac{\partial u_2}{\partial x_2} + l_2\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_1}{\partial x_1}\right), \\ \frac{dh_3}{dt} &= (1 - l_1 - l_2)\frac{\partial u_3}{\partial x_3} + l_3\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right). \end{aligned} \tag{3.9}$$

Let us now examine how the equation for the entropy will be in the case under consideration of strain along three axes under the selection made for the matrix β_{ij}. All the matrices σ_{ij}, c_{ij}, a_{ij}, $\dot{\varphi}_{ij}$ are diagonal for strains along three axes; hence, taking account of (1.12) the equation for the entropy is written in the form

$$\frac{dS}{dt} = \frac{1}{\rho E_S} \dot{\varphi}_i \sigma_i = \frac{1}{\rho E_S} (\dot{\varphi}_1\sigma_1 + \dot{\varphi}_2\sigma_2 + \dot{\varphi}_3\sigma_3).$$

The plastic terms $\dot{\varphi}_i$ are calculated by means of (3.3), therefore

$$\frac{dS}{dt} = \frac{1}{\rho E_S} \sigma_i \gamma_{ij} \frac{dh_j}{dt}.$$

Using (3.5), we obtain

$$\frac{dS}{dt} = \frac{1}{\rho E_S} \sigma_i \gamma_{ij} \sum_{\alpha=1}^{3} \beta_{j\alpha} \frac{\partial u_\alpha}{\partial x_\alpha}.$$

By definition

$$\|\gamma_{ij}\| = \|\beta_{ij}\|^{-1} - \|\delta_{ij}\|;$$

i.e.,

$$\frac{dS}{dt} = \frac{1}{\rho E_S} \sigma_i \sum_{\alpha=1}^{3} (\delta_{i\alpha} - \beta_{i\alpha}) \frac{\partial u_\alpha}{\partial x_\alpha}.$$

Using the formulas (3.7) and (3.8) selected for β_{ij}, we obtain an equation for S:

$$\rho E_S \frac{dS}{dt} = [(\sigma_1 - \sigma_2) l_2 + (\sigma_1 - \sigma_3) l_3] \frac{\partial u_1}{\partial x_1} + [(\sigma_2 - \sigma_3) l_3 + (\sigma_2 - \sigma_1) l_1] \frac{\partial u_2}{\partial x_2} + [(\sigma_3 - \sigma_1) l_1 + (\sigma_3 - \sigma_2) l_2] \frac{\partial u_3}{\partial x_3}. \quad (3.10)$$

Let us introduce terms describing the plastic strains in the equation for the tensor h_{ij} under arbitrary strains. We shall consider rotations of the elements of the medium not to be accompanied by plastic deformations since the selected plastic deformation characteristic h does not contain elements of the matrix U [see (3.2)]. We thereby set $\psi_{ij} = 0$ for $i \neq j$. We consider the diagonal elements of the matrix ψ_{ij} the same as in the case of strains along three axes:

$$\psi_{ii} = \gamma_{ij} dh_j/dt; \quad (3.11)$$

h_j are the principal values of the tensor h_{ij}.

Equations (2.4) and (2.5) take the form

$$\frac{dh_i}{dt} - \omega_{ii} = - \gamma_{ij} \frac{dh_j}{dt},$$
$$\left[U^* \frac{dU}{dt}\right]_{ij} - \frac{1}{g_i - g_j} (g_i \omega_{ij} + g_j \omega_{ji}) = 0, \quad i \neq j. \quad (3.12)$$

It can be shown that for such a choice of ψ_{ij}

$$\rho E_S \frac{dS}{dt} = \sum_{i=1}^{3} \sigma_i \psi_{ii} = \sigma_i \gamma_{ij} \frac{dh_j}{dt} \quad (3.13)$$

holds.

We also solve the equations for h_i for dh_i/dt. They can be obtained from (3.9) by replacing $\partial u_i/\partial x_i$ by ω_{ii}:

$$\frac{dh_1}{dt} = (1 - l_2 - l_3) \omega_{11} + l_1 (\omega_{22} + \omega_{33}),$$
$$\frac{dh_2}{dt} = (1 - l_3 - l_1) \omega_{22} + l_2 (\omega_{33} + \omega_{11}), \quad \frac{dh_3}{dt} = (1 - l_1 - l_2) \omega_{33} + l_3 (\omega_{11} + \omega_{22}). \quad (3.14)$$

Using (3.12) and (3.14), the equations for the tensor components h_{ij} can be written down. Again using the identity

$$U^* \frac{dH}{dt} U = \frac{dD_H}{dt} + U^* \frac{dU}{dt} D_H - D_H U^* \frac{dU}{dt},$$

we obtain

$$\frac{dH}{dt} = U \begin{Vmatrix} (1 - l_2 - l_3) \omega_{11} + l_1 (\omega_{22} + \omega_{33}) & -\frac{h_1 - h_2}{g_1 - g_2} (g_1 \omega_{12} + g_2 \omega_{21}) & -\frac{h_1 - h_3}{g_1 - g_3} (g_1 \omega_{13} + g_3 \omega_{31}) \\ -\frac{h_2 - h_1}{g_2 - g_1} (g_2 \omega_{21} + g_1 \omega_{12}) & (1 - l_3 - l_1) \omega_{22} + l_2 (\omega_{33} + \omega_{11}) & -\frac{h_2 - h_3}{g_2 - g_3} (g_2 \omega_{23} + g_3 \omega_{32}) \\ -\frac{h_3 - h_1}{g_3 - g_1} (g_3 \omega_{31} + g_1 \omega_{13}) & -\frac{h_3 - h_2}{g_3 - g_2} (g_3 \omega_{32} + g_2 \omega_{23}) & (1 - l_1 - l_2) \omega_{33} + l_3 (\omega_{11} + \omega_{22}) \end{Vmatrix} U^*. \quad (3.15)$$

Here $g_i = e^{-2h_i}$; $\|\omega_{ij}\| = U^* \left\| \frac{\partial u_n}{\partial x_m} \right\| U$.

An equation for the entropy S can be obtained from (3.15) for H and the energy conservation law. To do this, we use the formula for the stress

$$\|\sigma_{ij}\| = \rho \left\| \frac{\partial E}{\partial h_{ij}} \right\| = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} U^* = U D_\sigma U^*.$$

We have from the energy equation

$$\frac{dS}{dt} = \frac{1}{E_S}\frac{dE}{dt} - \frac{E_{h_{ij}}}{E_S}\frac{dh_{ij}}{dt} = \frac{\sigma_{ij}}{\rho E_S}\frac{\partial u_i}{\partial x_j} - \frac{1}{E_S}\, tr\left(\frac{\partial E}{\partial H}\frac{dH}{dt}\right) \tag{3.16}$$

and from (3.15) for H we have

$$tr\left(\frac{\partial E}{\partial H}\frac{dH}{dt}\right) = \frac{1}{\rho}\, [\sigma_1 [(1 - l_2 - l_3)\,\omega_{11} + l_1(\omega_{22} + \omega_{33})] +$$
$$+ \sigma_2 [(1 - l_3 - l_1)\,\omega_{22} + l_2(\omega_{33} + \omega_{11})] + \sigma_3 [(1 - l_1 - l_2)\,\omega_{33} + l_3(\omega_{11} + \omega_{22})]] =$$
$$= \frac{1}{\rho}\, [(1 - l_1 - l_2 - l_3)(\sigma_1\omega_{11} + \sigma_2\omega_{22} + \sigma_3\omega_{33}) + (l_1\sigma_1 + l_2\sigma_2 - l_3\sigma_3)(\omega_{11} + \omega_{22} + \omega_{33})].$$

It can be shown that

$$\sigma_1\omega_{11} + \sigma_2\omega_{22} + \sigma_3\omega_{33} = \sigma_{ij}\partial u_i/\partial x_j,$$
$$\omega_{11} + \omega_{22} + \omega_{33} = \partial u_i/\partial x_i.$$

Therefore,

$$tr\left(\frac{\partial E}{\partial H}\frac{dH}{dt}\right) = \frac{1}{\rho}(1 - l_1 - l_2 - l_3)\,\sigma_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}(l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right).$$

Substituting this expression into (3.16) we obtain

$$\rho E_S \frac{dS}{dt} = (l_1 + l_2 + l_3)\sigma_{ij}\frac{\partial u_i}{\partial x_j} - (l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right). \tag{3.17}$$

It can be seen that in the case of the diagonal matrix σ_{ij}, Eq. (3.17 agrees with (3.10) for the case of triaxial deformation.

Thus, (3.15) for H in combination with (1.1) and (1.2) for the velocities and energy for a known equation of state $E(h_1, h_2, h_3, S)$, by which the stresses (2.7) are determined, and the known coefficients $l_i(h_1, h_2, h_3, S)$ form a closed system of dynamics equations for an elastic–plastic medium. The entropy equation [(3.17)] can be included in the complete system in place of the energy equation.

4. CHARACTERISTICS

Let us study the characteristic surfaces of the system of equations formulated above. Just the realness of the characteristics is not sufficient for the system to be hyperbolic. Let us require that the matrix describing the sound wave propagation (we call it the acoustic matrix) be symmetric. This requirement can be satisfied because of a suitable selection of the parameters l_1, l_2, l_3 still not determined, after which the number of independent plasticity parameters is reduced to one: $L = l_1 + l_2 + l_3$. The symmetry of the acoustic matrix and the requirement that it be positive-definite automatically determine the realness of the characteristics, i.e., the hyperbolicity. Let us note that the symmetry property of the acoustic matrix holds for all elastic media (isotropic and anisotropic). Moreover, the symmetry is convenient for the construction of energy integrals, i.e., in obtaining a priori estimates of the solution and its derivatives.

We will evaluate the characteristics in a coordinate system coupled to the principal axes of the stress tensor. To do this, we shall write all the equations of the system in quasilinear form and shall set one in all the coefficients for the derivatives of the unknown functions for the matrix U. We obtain the following system of equations:

$$\begin{aligned}
\frac{du_i}{dt} &= \sum_{j=1}^{3}\left(E_{h_i h_j} - E_{h_i}\right)\frac{\partial h_{jj}}{\partial x_i} + E_{h_i S}\frac{\partial S}{\partial x_i} + \sum_{\substack{j=1\\ j\neq i}}^{3}\frac{E_{h_i} - E_{h_j}}{h_i - h_j}\frac{\partial h_{ij}}{\partial x_j},\\
\frac{dh_{11}}{dt} &= (1 - l_2 - l_3)\frac{\partial u_1}{\partial x_1} + l_1\frac{\partial u_2}{\partial x_2} + l_1\frac{\partial u_3}{\partial x_3},\\
\frac{dh_{22}}{dt} &= l_2\frac{\partial u_1}{\partial x_1} + (1 - l_3 - l_1)\frac{\partial u_2}{\partial x_2} + l_2\frac{\partial u_3}{\partial x_3},\\
\frac{dh_{33}}{dt} &= l_3\frac{\partial u_1}{\partial x_1} + l_3\frac{\partial u_2}{\partial x_2} + (1 - l_1 - l_2)\frac{\partial u_3}{\partial x_3},
\end{aligned} \tag{4.1}$$

$$\frac{dh_{ij}}{dt} = -\frac{h_i - h_j}{g_i - g_j}\left(g_i\frac{\partial u_i}{\partial x_j} + g_j\frac{\partial u_j}{\partial x_i}\right),\ i \neq j,$$

$$\frac{dS}{dt} = \frac{1}{\rho E_S}[(\sigma_1 - \sigma_2)l_2 + (\sigma_1 - \sigma_3)l_3]\frac{\partial u_1}{\partial x_1} + \frac{1}{\rho E_S}[(\sigma_2 - \sigma_3)l_3 + (\sigma_2 - \sigma_1)l_1]\frac{\partial u_2}{\partial x_2} + \frac{1}{\rho E_S}[(\sigma_3 - \sigma_1)l_1 + (\sigma_3 - \sigma_2)l_2]\frac{\partial u_3}{\partial x_3}.$$

Let us apply the method of [4, 5] to evaluate the characteristics of this system. By elimination, the differentiated system can be reduced to a system of second-order equations, closed in the highest derivatives with respect to the velocity. Let us note that the operation of differentiating the equations and reducing the equations to the principal axes are commutative to the accuracy of the lowest members, which are negligible for the evaluation of the characteristics.

Let us apply the operator $d/dt = \partial/\partial t + u_\alpha \partial/\partial x_\alpha$ to the equations for u_i in the system (4.1), and let us substitute the needed derivatives of dh_{ij}/dt and dS/dt with respect to the space coordinates x_i. Let us hence follow only the terms of the equations containing the highest second-order derivatives. We consequently obtain a system of equations which is closed in the highest terms in u_i:

$$\begin{aligned}\frac{d^2u_1}{dt^2} &= L_1\frac{\partial^2 u_1}{\partial x_1^2} + M_3 e^{-2h_2}\frac{\partial^2 u_1}{\partial x_2^2} + M_2 e^{-2h_3}\frac{\partial^2 u_1}{\partial x_3^2} + R_3\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + R_2\frac{\partial^2 u_3}{\partial x_1\partial x_3} + \ldots,\\ \frac{d^2u_2}{dt^2} &= P_3\frac{\partial^2 u_1}{\partial x_1\partial x_2} + M_3 e^{-2h_1}\frac{\partial^2 u_2}{\partial x_1^2} + L_2\frac{\partial^2 u_2}{\partial x_2^2} + M_1 e^{-2h_3}\frac{\partial^2 u_2}{\partial x_3^2} + R_1\frac{\partial^2 u_3}{\partial x_1\partial x_3} + \ldots,\\ \frac{d^2u_3}{dt^2} &= P_2\frac{\partial^2 u_1}{\partial x_1\partial x_3} + P_1\frac{\partial^2 u_2}{\partial x_2\partial x_3} + M_2 e^{-2h_1}\frac{\partial^2 u_3}{\partial x_1^2} + M_1 e^{-2h_2}\frac{\partial^2 u_3}{\partial x_2^2} + L_3\frac{\partial^2 u_3}{\partial x_3^2} + \ldots.\end{aligned} \tag{4.2}$$

The coefficients L_i, M_i, R_i, P_i are evaluated from the formulas

$$L_1 = (E_{h_1h_1} - E_{h_1}) + (E_{h_1h_2} - E_{h_1h_1})l_2 + (E_{h_1h_3} - E_{h_1h_1})l_3 + \frac{E_{h_1S}}{\rho E_S}[(\sigma_1 - \sigma_2)l_2 + (\sigma_1 - \sigma_3)l_3],$$

$$L_2 = (E_{h_2h_2} - E_{h_2}) + (E_{h_2h_3} - E_{h_2h_2})l_3 + (E_{h_2h_1} - E_{h_2h_2})l_1 + \frac{E_{h_2S}}{\rho E_S}[(\sigma_2 - \sigma_3)l_3 + (\sigma_2 - \sigma_1)l_1],$$

$$L_3 = (E_{h_3h_3} - E_{h_3}) + (E_{h_3h_1} - E_{h_3h_3})l_1 + (E_{h_3h_2} - E_{h_3h_3})l_2 + \frac{E_{h_3S}}{\rho E_S}[(\sigma_3 - \sigma_1)l_1 + (\sigma_3 - \sigma_2)l_2],$$

$$M_1 = \frac{E_{h_2} - E_{h_3}}{e^{2h_2} - e^{2h_3}},\quad M_2 = \frac{E_{h_3} - E_{h_1}}{e^{2h_3} - e^{2h_1}},\quad M_3 = \frac{E_{h_1} - E_{h_2}}{e^{2h_1} - e^{2h_2}},$$

$$R_i = E_{h_jh_k} + \frac{e^{2h_k}E_{h_j} - e^{2h_j}E_{h_k}}{e^{2h_j} - e^{2h_k}} + (E_{h_jh_i} - E_{h_jh_k})l_i + (E_{h_jh_j} - E_{h_jh_k})l_j + \frac{E_{h_jS}}{\rho E_S}[(\sigma_k - \sigma_i)l_i + (\sigma_k - \sigma_j)l_j],$$

$$P_i = E_{h_jh_k} + \frac{e^{2h_k}E_{h_j} - e^{2h_j}E_{h_k}}{e^{2h_j} - e^{2h_k}} + (E_{h_kh_k} - E_{h_kh_j})l_k + (E_{h_kh_i} - E_{h_kh_j})l_i + \frac{E_{h_kS}}{\rho E_S}[(\sigma_j - \sigma_k)l_k + (\sigma_j - \sigma_i)l_i],$$

$$i, j, k = 1, 2, 3;\ i \neq j \neq k.$$

Let us note that the equations for h_{ij}, S cannot be considered; their corresponding characteristics correspond to streamlines [4].

If $(\omega, \xi_1, \xi_2, \xi_3)$ denotes the vector normal to the characteristic surface, the equation of the characteristic normals for the system (4.2) has the form

$$\det(\Omega^2 I - \Lambda) = 0, \tag{4.3}$$

where $\Omega = \omega + u_\alpha \xi_\alpha$ and Λ is an acoustic matrix of the form

$$\Lambda = \begin{Vmatrix} L_1\xi_1^2 + M_3e^{2h_2}\xi_2^2 + M_2e^{2h_3}\xi_3^2 & R_3\xi_1\xi_2 & R_2\xi_1\xi_3 \\ P_3\xi_2\xi_1 & M_3e^{2h_1}\xi_1^2 + L_2\xi_2^2 + M_1e^{2h_3}\xi_3^2 & R_1\xi_2\xi_3 \\ P_2\xi_3\xi_1 & P_1\xi_3\xi_2 & M_2e^{2h_1}\xi_1^2 + M_1e^{2h_2}\xi_2^2 + L_3\xi_3^2 \end{Vmatrix}. \tag{4.4}$$

For the system (4.2) to be hyperbolic it is necessary to require that (4.3) have positive roots Ω^2. For this matrix Λ, it is difficult to make a conclusion about the allowable domains of the coefficients L_i, M_i, R_i, P_i in which (4.3) has the root $\Omega^2 > 0$. There is still freedom in the choice of the coefficients l_1, l_2, l_3 governing the plastic deformation and entering in L_i, M_i, R_i, P_i. Making use of this circumstance, let us require that the matrix Λ be symmetric. If the matrix Λ is symmetric, then under the condition of its positive definiteness we at once obtain $\Omega^2 > 0$. Finally, it is also difficult to write down all the conditions for the positive definiteness of such a matrix Λ, but certain necessary conditions for positive definiteness can at least be mentioned. The requirement of symmetry of the matrix Λ reduces to satisfying the equalities $P_1=R_1$, $P_2=R_2$, $P_3=R_3$, which are homogeneous linear equations for l_1, l_2, l_3:

$$(E_{h_2h_1} - E_{h_3h_1})\, l_1 + (E_{h_2h_2} - E_{h_2h_3})\, l_2 + (E_{h_3h_2} - E_{h_3h_3})\, l_3 +$$

$$+ \frac{1}{\rho E_S}[E_{h_2S}(\sigma_3 - \sigma_1) + E_{h_3S}(\sigma_1 - \sigma_2)]\, l_1 + \frac{E_{h_2S}}{\rho E_S}(\sigma_3 - \sigma_2)\, l_2 + \frac{E_{h_3S}}{\rho E_S}(\sigma_3 - \sigma_2)\, l_3 = 0,$$

$$(E_{h_1h_3} - E_{h_1h_1})\, l_1 + (E_{h_3h_2} - E_{h_1h_2})\, l_2 + (E_{h_3h_3} - E_{h_3h_1})\, l_3 + \qquad (4.5)$$

$$+ \frac{E_{h_1S}}{\rho E_S}(\sigma_1 - \sigma_3)\, l_1 + \frac{1}{\rho E_S}[E_{h_1S}(\sigma_2 - \sigma_3) + E_{h_3S}(\sigma_1 - \sigma_2)]\, l_2 + \frac{E_{h_1S}}{\rho E_S}(\sigma_1 - \sigma_3)\, l_3 = 0,$$

$$(E_{h_1h_1} - E_{h_1h_2})\, l_1 + (E_{h_1h_2} - E_{h_2h_2})\, l_2 + (E_{h_1h_3} - E_{h_2h_3})\, l_3 +$$

$$+ \frac{E_{h_1S}}{\rho E_S}(\sigma_2 - \sigma_1)\, l_1 + \frac{E_{h_2S}}{\rho E_S}(\sigma_2 - \sigma_1)\, l_2 + \frac{1}{\rho E_S}[E_{h_1S}(\sigma_2 - \sigma_3) + E_{h_2S}(\sigma_3 - \sigma_1)]\, l_3 = 0.$$

Let us assume the equation of state to have the form

$$E(h_1, h_2, h_3, S) = E^0(\rho, S) + E^1(h_1, h_2, h_3).$$

For such equations of state,

$$(E_S)_{h_1} = (E_S)_{h_2} = (E_S)_{h_3} = -\rho E^0_{\rho S}(\rho, S) \qquad (4.6)$$

holds. By using (4.6) it is seen that the sum of the left sides of (4.5) is identically zero. Therefore, the system (4.5) is solvable and the parameters l_1, l_2, l_3 can be selected in terms of one arbitrary parameter L. Let $L = l_1 + l_2 + l_3$ and taking into account that $E_S = T$ is the temperature, $(E_S)_{h_i} = -\rho E^0_{\rho S} = -\rho T_\rho$, we obtain

$$l_1 = \frac{L}{F_1 + F_2 + F_3}\Big[F_1 + \frac{T_\rho}{T}[\sigma_1(2E_{h_3h_2} - E_{h_2h_2} - E_{h_3h_3}) +$$

$$+ \sigma_2(E_{h_1h_2} + E_{h_3h_3} - E_{h_3h_1} - E_{h_3h_2}) + \sigma_3(E_{h_1h_3} - E_{h_2h_2} - E_{h_2h_1} - E_{h_2h_3})]\Big],$$

$$l_2 = \frac{L}{F_1 + F_2 + F_3}\Big[F_2 + \frac{T_\rho}{T}[\sigma_1(E_{h_2h_1} + E_{h_3h_3} - E_{h_3h_1} - E_{h_3h_2}) + \qquad (4.7)$$

$$+ \sigma_2(2E_{h_1h_3} - E_{h_1h_1} - E_{h_3h_3}) + \sigma_3(E_{h_2h_3} + E_{h_1h_1} - E_{h_2h_1} - E_{h_1h_3})]\Big],$$

$$l_3 = \frac{L}{F_1 + F_2 + F_3}\Big[F_3 + \frac{T_\rho}{T}[\sigma_1(E_{h_3h_1} + E_{h_2h_2} - E_{h_2h_3} - E_{h_2h_1}) +$$

$$+ \sigma_2(E_{h_3h_3} + E_{h_1h_1} - E_{h_1h_3} - E_{h_1h_2}) + \sigma_3(2E_{h_1h_2} - E_{h_1h_1} - E_{h_2h_2})]\Big],$$

where F_i is the sum of the cofactors of elements of the i-th column (row) of the matrix $\|E_{h_ih_j}\|$:

$$F_1 = \begin{vmatrix} E_{h_2h_2} & E_{h_2h_3} \\ E_{h_3h_2} & E_{h_3h_3} \end{vmatrix} - \begin{vmatrix} E_{h_1h_2} & E_{h_1h_3} \\ E_{h_3h_2} & E_{h_3h_3} \end{vmatrix} + \begin{vmatrix} E_{h_1h_2} & E_{h_1h_3} \\ E_{h_2h_2} & E_{h_2h_3} \end{vmatrix},$$

$$F_2 = \begin{vmatrix} E_{h_1h_1} & F_{h_1h_3} \\ E_{h_3h_1} & E_{h_3h_3} \end{vmatrix} - \begin{vmatrix} E_{h_2h_1} & E_{h_2h_3} \\ E_{h_3h_1} & E_{h_3h_3} \end{vmatrix} - \begin{vmatrix} E_{h_1h_1} & E_{h_1h_3} \\ E_{h_2h_1} & E_{h_2h_3} \end{vmatrix},$$

$$F_3 = \begin{vmatrix} E_{h_1h_1} & E_{h_1h_2} \\ E_{h_2h_1} & E_{h_2h_2} \end{vmatrix} - \begin{vmatrix} E_{h_1h_1} & E_{h_1h_2} \\ E_{h_3h_1} & E_{h_3h_2} \end{vmatrix} + \begin{vmatrix} E_{h_2h_1} & E_{h_3h_2} \\ E_{h_1h_3} & E_{h_3h_3} \end{vmatrix}.$$

Thus, the parameters l_1, l_2, l_3, expressed in terms of derivatives of the equation of state and one arbitrary plasticity parameter L have been selected such that the acoustic matrix (4.4) would become symmetric:

$$\Lambda = \begin{pmatrix} L_1\xi_1^2 + M_3 e^{2h_2}\xi_2^2 + M_2 e^{2h_3}\xi_3^2 & P_3\xi_1\xi_2 & P_2\xi_1\xi_3 \\ P_3\xi_2\xi_1 & M_3 e^{2h_1}\xi_1^2 + L_2\xi_2^2 + M_1 e^{2h_3}\xi_3^2 & P_1\xi_2\xi_3 \\ P_2\xi_3\xi_1 & P_1\xi_3\xi_2 & M_2 e^{2h_1}\xi_1^2 + M_1 e^{2h_2}\xi_2^2 + L_3\xi_3^2 \end{pmatrix}.$$

The hyperbolicity conditions now reduce to the requirement of positive definiteness of Λ. Let us write some necessary requirements for the positive definiteness

$$L_1 > 0,\ L_2 > 0,\ L_3 > 0,\ M_1 > 0,\ M_2 > 0,\ M_3 > 0.$$

Let us note that for $L=0$ (i.e., for $l_1=l_2=l_3=0$) the matrix Λ is an acoustic matrix of the nonlinear elasticity theory equations which were studied in [3, 4].

The plasticity parameter L can depend on all the stress tensor invariants and on the temperature.

The requirement of positive definiteness of the matrix Λ reduces to constraints on the equation of state $E(h_1, h_2, h_3, S)$ and the plasticity parameter $L(h_1, h_2, h_3, S)$. Unfortunately, it is difficult to write down explicitly convenient inequalities describing these constraints.

As an illustration, let us determine the parameters l_1, l_2, l_3 for certain specific equations of state. For instance, let us consider an equation of state of the form (it is studied in [6] for a number of metals)

$$E(h_1, h_2, h_3, S) = E^0(\rho, S) + 2B(\rho)D, \tag{4.8}$$

where $\rho=\rho^0 e^{-(h_1+h_2+h_3)}$; $D=\frac{1}{2}\sum_{i=1}^{3}\left(h_i-\frac{h_1+h_2+h_3}{3}\right)^2$; the function $B(\rho)$ has the meaning of the square of the velocity of transverse sound wave propagation. Let us use Eqs. (4.5), which for the equation of state state (4.8) $[T = E^0_S(\rho, S)]$ become

$$l_2 - l_3 = \left(\frac{T_\rho}{T} - \frac{B_\rho}{B}\right)\frac{\sigma_3-\sigma_2}{2B}(l_1+l_2+l_3),$$
$$l_3 - l_1 = \left(\frac{T_\rho}{T} - \frac{B_\rho}{B}\right)\frac{\sigma_1-\sigma_3}{2B}(l_1+l_2+l_3),$$
$$l_1 - l_2 = \left(\frac{T_\rho}{T} - \frac{B_\rho}{B}\right)\frac{\sigma_2-\sigma_1}{2B}(l_1+l_2+l_3),$$

where $\sigma_i = -\rho^2 E^0_\rho(\rho, S) - 2\rho^2 B_\rho(\rho) D + 2\rho B(\rho)\left(h_i - \frac{h_1+h_2+h_3}{3}\right)$. Using the notation $L = l_1 + l_2 + l_3$, we obtain

$$l_i = \frac{L}{3}\left[1 - \frac{3}{2B}\left(\frac{T_\rho}{T} - \frac{B_\rho}{B}\right)\left(\sigma_i - \frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)\right],\quad i = 1, 2, 3. \tag{4.9}$$

Now, let us write the equation of the characteristic normals for the equation of state,

$$E = \frac{\lambda}{2\rho_0}(h_1+h_2+h_3)^2 + \frac{\mu}{\rho_0}(h_1^2+h_2^2+h_3^2),$$

where we calculate all the coefficients of the acoustic matrix Λ at the point $h_1=h_2=h_3=0$. Such an acoustic matrix will describe sound wave propagation in a "linearly plastic" medium.

Using the formulas to evaluate L_i, M_i, P_i and taking into account that $l_1=l_2=l_3=L/3$ in the case under consideration, we have

$$L_1 = L_2 = L_3 = \frac{1}{\rho_0}\left(\lambda + 2\mu - \frac{4}{3}\mu L\right),\quad M_1 = M_2 = M_3 = \frac{\mu}{\rho_0},$$
$$P_1 = P_2 = P_3 = \frac{1}{\rho_0}\left(\lambda + \mu + \frac{2}{3}\mu L\right).$$

The matrix Λ will have the form

$$\rho_0\Lambda = \left\|\begin{matrix} \left(\lambda+2\mu-\frac{4}{3}\mu L\right)\xi_1^2+\mu(\xi_2^2+\xi_3^2) & \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_1\xi_2 & \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_1\xi_2 \\ \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_2\xi_1 & \left(\lambda+2\mu-\frac{4}{3}\mu L\right)\xi_2^2+\mu(\xi_3^2+\xi_1^2) & \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_1\xi_3 \\ \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_3\xi_1 & \left(\lambda+\mu+\frac{2}{3}\mu L\right)\xi_3\xi_2 & \left(\lambda+2\mu-\frac{4}{3}\mu L\right)\xi_3^2+\mu(\xi_1^2+\xi_2^2) \end{matrix}\right\|.$$

Let us examine the case of plane waves, which corresponds to a two-dimensional system of equations. For this case the second-order matrix Λ with $\xi_3=0$ must be studied. The characteristic equation [(4.3)] will have the form (we use the notation $K = \lambda + \frac{2}{3}\mu$)

$$(\rho_0\Omega^2)^2 - \left(K + \mu + \frac{4}{3}\mu(1-L)\right)(\xi_1^2 + \xi_2^2)(\rho_0\Omega^2) + \mu\left(K + \frac{4}{3}\mu(1-L)\right)\times$$
$$\times(\xi_1^4 + \xi_2^4) - 2\mu\left(K - 2\left(K + \frac{1}{3}\mu\right)(1-L) + \frac{2}{3}\mu(1-L)^2\right)\xi_1^2\xi_2^2 = 0.$$

A section of the surfaces of the characteristic normals to the plane Ω = const is shown in Fig. 1.

Let us note that a detailed study of the characteristic normal surfaces and the characteristic surfaces for anisotropic elastic media in the case of plane waves is contained in [7].

5. PLASTIC DEFORMATIONS. ENTROPY GROWTH

The requirement of symmetry of the acoustic matrix for a certain class of equations of state permitted extraction of a parameter L characterizing the plastic deformations. Let us now examine how the plastic deformations are determined by using this parameter L.

The elastic–plastic strain equations for effective elastic strains have the form

$$dh_i/dt - \omega_{ii} = -\psi_{ii} = -\gamma_{ij}dh_j/dt.$$

The coefficients γ_{ij} are determined from the formula

$$||\gamma_{ij}|| = ||\beta_{ij}||^{-1} - ||\delta_{ij}||,$$

but the β_{ij} are expressed in terms of the three parameters l_1, l_2, l_3 in conformity with (3.7) and (3.8), which are, in turn, expressed in terms of the one parameter $L = l_1 + l_2 + l_3$ from the condition of symmetry of the acoustic matrix according to (4.7).

Starting from this, the coefficients γ_{ij} can be evaluated by means of the formulas

$$\begin{gathered}\gamma_{11} = (L - l_1)/(1 - L),\ \gamma_{22} = (L - l_2)/(1 - L),\\ \gamma_{33} = (L - l_3)/(1 - L),\\ \gamma_{12} = \gamma_{13} = -l_1/(1 - L),\ \gamma_{21} = \gamma_{23} = -l_2/(1 - L),\\ \gamma_{31} = \gamma_{32} = -l_3/(1 - L),\end{gathered} \tag{5.1}$$

from which we get

$$\psi_{ii} = \gamma_{ij}\frac{dh_j}{dt} = \frac{1}{1-L}\left[L\frac{dh_i}{dt} - l_i\frac{d(h_1 + h_2 + h_3)}{dt}\right], \tag{5.2}$$

which expresses the rate of change of the plastic strains in terms of the rate of change of the effective elastic strains, meaning in terms of the rate of change of the stress.

Indeed, in the case of triaxial strain without rotation, we have

$$\frac{dh_i^p}{dt} = \psi_{ii} = \frac{1}{1-L}\left[L\frac{dh_i}{dt} - l_i\frac{d(h_1 + h_2 + h_3)}{dt}\right]. \tag{5.3}$$

Using (4.9) for l_i for the equation of state (4.8), and for

$$\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \rho B(\rho)\left(h_i - \frac{h_1 + h_2 + h_3}{3}\right),$$

we obtain

$$\frac{dh_i^p}{dt} = \frac{L}{1-L}\frac{1}{2B}\frac{d}{dt}\frac{1}{\rho}\left(\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right) - \frac{L}{1-L}\frac{T_\rho}{2\rho BT}\left(\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)\frac{d\rho}{dt}. \tag{5.4}$$

It can be seen that if the temperature effects are not taken into account, the relationships (5.3) will take the form

$$\frac{dh_i^p}{dt} = \frac{L}{1-L}\frac{1}{2B}\frac{d}{dt}\frac{1}{\rho}\left(\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right).$$

These relationships, which are a particular case of (5.2), recall the equations of the Hencky deformation theory of plasticity [8]. Hence, the possible limit cases can be seen: If $L = 0$, then $dh_i^p/dt = 0$; i.e., the strains occur

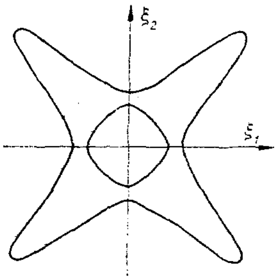

Fig. 1

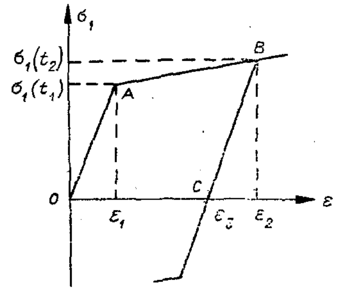

Fig. 2

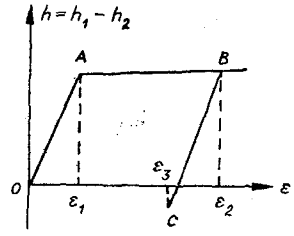

Fig. 3

elastically; if L = 1, then $\frac{d}{dt}\frac{1}{\rho}\left(\sigma_i - \frac{\sigma_1+\sigma_2+\sigma_3}{3}\right) = 0$; i.e., the plastic strains proceed with an unchanged stress deviator (ideal plasticity).

The domain of variation of the plasticity parameter L is in the range [0, 1]:

$$0 \leqslant L \leqslant 1.$$

Now, let us examine the equation for the entropy. The entropy growth law affords a possibility of extracting the domain where the strains occur only elastically. The equation for the entropy has the form (3.13). Using (5.1) for ψ_{ii}, we obtain

$$\rho E_S \frac{dS}{dt} = \frac{L}{1-L}\sigma_i \frac{d\left(h_i - \frac{h_1+h_2+h_3}{3}\right)}{dt} + \sigma_i\left(\frac{L}{3} - l_i\right)\frac{d(h_1+h_2+h_3)}{dt}. \tag{5.5}$$

The requirement of a nondecrease in entropy implies the necessity to comply with the inequality

$$\frac{1}{1-L}[(L-l_1)\sigma_1 - l_2\sigma_2 - l_3\sigma_3]\frac{dh_1}{dt} + \frac{1}{1-L}[(L-l_2)\sigma_2 - l_3\sigma_3 - l_1\sigma_1]\frac{dh_2}{dt} + \frac{1}{1-L}[(L-l_3)\sigma_3 - l_1\sigma_1 - l_2\sigma_2]\frac{dh_3}{dt} \geqslant 0. \tag{5.6}$$

As soon as this inequality is spoiled, it is necessary to set L = 0 (this means $l_i = 0$) even if the intensity of the tangential strains $h > h_*$ is the yield stress.

6. ELASTIC – PLASTIC STRAINS OF A PLANE LAYER

As an illustration of the use of the system of equations we compiled, let us consider the problem of strain of a flat layer. The equation to describe such processes by using the Maxwell model is presented in [3].

To derive the plane layer strain equations by means of the elastic–plastic model, we write down a one-dimensional system of equations. We use (1.1), (3.15), (3.17) for the velocity, the strain tensor, and the entropy. Since the motion is one-dimensional, we set $u_1 = u$, $u_2 = u_3 = 0$, $h_{11} = h_1$, $h_{22} = h_{33} = h_2$, $h_{ij} = 0$ $(i \neq j)$; hence $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_{33} = \sigma_2$, $l_2 = l_3$, and we consider that the unknown functions depend only on one space coordinate $x = x_1$. The one-dimensional equations have the form

$$\frac{du}{dt} - \frac{1}{\rho}\frac{\partial\sigma_1}{\partial x} = 0, \quad \frac{dh_1}{dt} - (1-2l_2)\frac{\partial u}{\partial x} = 0,$$
$$\frac{dh_2}{dt} - l_2\frac{\partial u}{\partial x} = 0, \quad \frac{dS}{dt} - \frac{2l_2(\sigma_1 - \sigma_2)}{\rho E_S}\frac{\partial u}{\partial x} = 0.$$

For simplicity we shall study processes without taking account of temperature effects by assuming that the stresses are associated with the effective elastic strain by relationships of the type of Hooke's law:

$$\sigma_i = \frac{\lambda}{\rho_0}(h_1+h_2+h_3) + \frac{2\mu}{\rho_0}h_i.$$

Hence, we shall not consider the equation for the entropy. Moreover, $l_1 = l_2 = l_3 = L/3$ holds for such a dependence $\sigma_i(h_1, h_2, h_3)$. Furthermore, small time intervals are considered, for which it can be considered that $\partial u/\partial x = \dot{\varepsilon} = \text{const}$ (it hence follows that the equation for u cannot also be considered).

The final system of equations describing the strain of a plane layer becomes (after the simplifications are made):

$$\frac{dh_1}{dt} = \left(1 - \frac{2}{3}L\right)\dot{\varepsilon}, \quad \frac{dh_2}{dt} = \frac{1}{3}L\dot{\varepsilon}. \tag{6.1}$$

The stresses are evaluated by the formulas

$$\sigma_1 = \frac{\lambda + 2\mu}{\rho_0}h_1 + \frac{2\lambda}{\rho_0}h_2, \quad \sigma_2 = \frac{\lambda}{\rho_0}h_1 + \frac{2(\lambda+\mu)}{\rho_0}h_2. \tag{6.2}$$

The inequality (5.5), which has, for the case under consideration, the form

$$\frac{L}{1-L}\sigma_i \frac{d\left(h_i - \frac{h_1+h_2+h_3}{3}\right)}{dt} = \frac{L}{1-L}\left(\sigma_i - \frac{\sigma_1+\sigma_2+\sigma_3}{3}\right) \geqslant \frac{d\left(h_i - \frac{h_1+h_2+h_3}{3}\right)}{dt} \geqslant 0$$

or

$$\frac{\mu}{\rho_0}\frac{L}{1-L}\frac{d}{dt}\sum_{i=1}^{3}\left(h_i - \frac{h_1+h_2+h_3}{3}\right)^2 = \frac{\rho_0}{\mu}\frac{L}{1-L}\frac{d}{dt}\sum_{i=1}^{3}\left(\sigma_i - \frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)^2 \geqslant 0,$$

remains, as before, the criterion for the progress of the plastic strains. In our case $L=0$, therefore, if at least one of the inequalities

$$h = |h_1 - h_2| < h_*, \quad \frac{d}{dt}|h_1 - h_2| < 0 \tag{6.3}$$

is satisfied. We consider $L = \text{const} > 0$ outside these domains of variation of $h_1 - h_2$.

Therefore, the system (6.1) describing the strain of a plane layer is closed by using the selection of two constants h_* and L, which characterize the plastic properties of the medium. Let us note that the quantity $\varepsilon = \dot{\varepsilon}t$ has the meaning of the real layer strain. Indeed, the total strain is determined from (5.2)

$$\frac{dh_1}{dt} + \frac{dh_1^p}{dt} = \frac{dh_1}{dt} + \frac{1}{1-L}\left[L\frac{dh_1}{dt} - \frac{L}{3}\frac{d(h_1+h_2+h_3)}{dt}\right] =$$

$$= \frac{1}{1-L}\frac{dh_1}{dt} - \frac{L}{3(1-L)}\frac{d(h_1+2h_2)}{dt} = \frac{1-\frac{2}{3}L}{1-L}\dot{\varepsilon} - \frac{L}{3(1-L)}\dot{\varepsilon} = \dot{\varepsilon},$$

from which $h_1 + h_1^p = \dot{\varepsilon}t + \text{const}$.

Let us start to stretch a layer at the constant rate of deformation $\dot{\varepsilon} > 0$. Since $|h_1 - h_2| < h_*$, we have ($L = 0$)

$$h_1(t) = \dot{\varepsilon}t, \; h_2(t) = 0, \; \sigma_1(t) = [(\lambda + 2\mu)/\rho_0]\dot{\varepsilon}t.$$

As soon as $h_1 - h_2 = h_*$, we consider $L = \text{const} > 0$. Let us denote this time by t_1:

$$h_1(t_1) = \dot{\varepsilon}t_1 = \varepsilon_1 = h_*, \; \sigma_1(t_1) = [(\lambda + 2\mu)/\rho_0]h_*.$$

Let us continue to stretch the layer further until the time t_2; hence

$$h_1(t) = h_* + \left(1 - \frac{2}{3}L\right)\dot{\varepsilon}(t - t_1), \quad h_2(t) = \frac{1}{3}L\dot{\varepsilon}(t - t_1),$$

$$\sigma_1(t) = \sigma_1(t_1) + \frac{\lambda + 2\mu - \frac{4}{3}\mu L}{\rho_0}\dot{\varepsilon}(t - t_1), \tag{6.4}$$

$$h_1(t) - h_2(t) = h_* + (1 - L)\dot{\varepsilon}(t - t_1).$$

It is seen that σ_1 grows for strains beyond the elastic limit, and the tangential strain intensity $h = |h_1 - h_2| > h_*$ also grows (Figs. 2 and 3, sections AB). Let us select some time, starting from which the tension of the rod ceases and we start to compress it at the strain rate $-\dot{\varepsilon}$ [in place of $\dot{\varepsilon}$ in (6.1), we must substitute $-\dot{\varepsilon}$]. By virtue of conditions (6.3), we set $L = 0$ on this section of the strain. Solving (6.1), we find on the section BC

$$h_1(t) = h_1(t_2) - \dot{\varepsilon}(t - t_2), \; h_2(t) = h_2(t_2),$$

$$\sigma_1(t) = \sigma_1(t_2) - [(\lambda + 2\mu)/\rho_0]\dot{\varepsilon}(t - t_2),$$

$$h_1(t) - h_2(t) = h_1(t_2) - h_2(t_2) - \dot{\varepsilon}(t - t_2),$$

where $h_1(t_2)$, $h_2(t_2)$, $\sigma_1(t_2)$ are determined from (6.4). Let us compress the layer until $\sigma_1 = 0$. Denoting this time by t_3, we determine it from the equation

$$\sigma_1(t_3) = \sigma_1(t_2) - [(\lambda + 2\mu)/\rho_0]\dot{\varepsilon}(t_3 - t_2) = 0.$$

We find

$$\dot{\varepsilon}(t_3 - t_2) = [\rho_0/(\lambda + 2\mu)]\sigma_1(t_2).$$

At the time t_3

$$\sigma_1(t_3) = 0,\ h_1(t_3) = h_1(t_2) - \dot{\varepsilon}(t_3 - t_2),\ h_2(t_3) = h_2(t_2).$$

Furthermore, at this time

$$h_1(t_3) - h_2(t_3) = h_1(t_2) - h_2(t_2) - \dot{\varepsilon}(t_3 - t_2) =$$
$$= h_* + \left(1 - \frac{2}{3}L\right)\dot{\varepsilon}(t_2 - t_1) - \frac{1}{3}L\dot{\varepsilon}(t_2 - t_1) - \dot{\varepsilon}(t_3 - t_2) = -\frac{\left(\lambda + \frac{2}{3}\mu\right)L}{\lambda + 2\mu}\dot{\varepsilon}(t_2 - t_1),$$

and since, $L > 0$, $\lambda + 2\mu/3 > 0$, $\lambda + 2\mu > 0$, we have $h_1(t_3) - h_2(t_3) < 0$. Therefore, when $\sigma_1 = 0$, the tangential strain intensity is $h_1 - h_2 < 0$.

Let us note the following:

1. If a rod is again subjected to tension at the same strain rate $\dot{\varepsilon}$, then a larger strain than $\varepsilon_1 = h_*$ must be performed to emerge in the plasticity domain $|h_1 - h_2| \geq h_*$ since we start the tension at $h_1 - h_2 < 0$. It is hence evident that $\sigma_1 = \sigma_1(\varepsilon_2)$ upon arrival at the plastic section (see Fig. 2). This effect interprets the hardening phenomenon.

2. If we continue to compress the rod further from the time $\sigma_1 = 0$ at the same strain rate, then it is necessary to perform a lesser deformation to emerge in the plasticity domain for the reason that $h_1 - h_2 < 0$ already, where $\sigma_1 < \sigma_1(\varepsilon_1)$ upon the emergence into plasticity (see Fig. 2). This phenomenon has the character of the Bauschinger effect.

If the nonlinear dependence $\sigma_i = \sigma_i(h_1, h_2, h_3, S)$, $l_i = l_i(h_1, h_2, h_3, S)$ is considered with the temperature taken into account, then all the strain sections in Figs. 2 and 3 will differ from the rectilinear.

The author is grateful to S. K. Godunov for interest in the research.

LITERATURE CITED

1. Kh. A. Rakhmatulin, "On unloading wave propagation," Prikl. Mat. Mekh., 9, No. 1, 91-100 (1945).
2. Kh. A. Rakhmatulin and Yu. A. Dem'yanov, Strength under Intensive Short-Range Loads [in Russian], Fizmatgiz, Moscow (1961).
3. S. K. Godunov, Elements of the Mechanics of a Continuous Medium [in Russian], Nauka, Moscow (1978).
4. S. K. Godunov and E. I. Romenskii, "Nonstationary equations of the nonlinear theory of elasticity in Euler coordinates," Zh. Prikl. Mekh. Tekh. Fiz., No. 6, 124-144 (1972).
5. E. I. Romenskii, "Cones of characteristics of the nonlinear elasticity theory equations," Zh. Prikl. Mekh. Tekh. Fiz., No. 3, 126-132 (1974).
6. S. K. Godunov, N. S. Kozin, and E. I. Romenskii, "Equation of state of the elastic energy of metals for a nonglobal strain tensor," Zh. Prikl. Mekh. Tekh. Fiz., No. 2, 123-128 (1974).
7. R. G. Payton, "Two-dimensional wavefront shape induced in a homogeneously strained elastic body by a point perturbing body force," Arch. Rat. Mech. Anal., 32, No. 4, 311-330 (1969).
8. L. M. Kachanov, Principles of Plasticity Theory [in Russian], Nauka, Moscow (1969).